Kvantne statistike i ansambli

Razmatramo neinteragujuće sisteme kvantnih čestica (identičnost)

- Kvantni ansambli

MKA
KA
VKA $\longrightarrow$ relevantan jer je najopštiji

- Kvantne statistike : $\mathcal{H} = \mathcal{H}^0 \otimes \mathcal{H}^S$

Bose - Einsteinova (celobrojni spin)

Fermi - Diracova (polucelobrojni spin)

- Pojam statističkog operatora

$\hat{\rho}$:
a) $\hat{\rho} > 0$
b) $\hat{\rho}^+ = \hat{\rho}$
c) $tr\hat{\rho} = 1$

$$\hat{\rho} = \sum_i w_i |\psi_i\rangle\langle\psi_i|$$

$$\hat{\rho} = \frac{e^{-\beta(\hat{H}_N - \mu\hat{N})}}{\boxed{\Xi}} , \quad W_{N,n} = \frac{e^{-\beta(E_{N,n} - \mu N)}}{\boxed{\Xi}}$$

$\boxed{\text{VKA}}$

$$\boxed{\Xi} = \sum_{N=0}^{\infty}\sum_n e^{-\beta(E_{N,n} - \mu N)} \equiv tr\, e^{-\beta(\hat{H}_N - \mu\hat{N})}$$

$$W_n = \frac{e^{-\beta E_n}}{Z} , \quad Z = \sum_n e^{-\beta E_n}$$

$\boxed{\text{KA}}$

$\longrightarrow$ sumiranje po stanjima

- Srednja vrednost opservable

$$\langle \hat{A} \rangle = \mathrm{tr}\, \hat{A}\hat{\rho}$$

VKA : $\langle \hat{A}_N \rangle = \frac{1}{\Xi} \mathrm{tr}\left(e^{-\beta(\hat{\mathcal{H}}_N - \mu \hat{N})} \hat{A}_N \right)$

KA : $\langle \hat{A} \rangle = \frac{1}{Z} \mathrm{tr}\left(e^{-\beta \hat{\mathcal{H}}} \hat{A} \right)$

- Sistem nezavisnih čestica

$$\begin{pmatrix} n_1 & n_2 & \dots & n_f \dots \\ \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_f \dots \end{pmatrix}$$ - raspodela čestica po jednočestičnim energijama

$$\sum_f n_f = N \qquad \sum_f n_f \varepsilon_f = E_{\{n_f\}}$$

$n_f = 0, 1, 2, \dots$ BA statistika

$n_f = 0, 1$ FD statistika

$$\Xi = \prod_f \left(1 \pm e^{-\beta(\varepsilon_f - \mu)} \right)^{\pm 1} \qquad \begin{matrix} + \text{FD} \\ - \text{BA} \end{matrix}$$

$$\Omega = -kT \ln \Xi$$

$$S = -\left(\frac{\partial \Omega}{\partial T} \right)_{V,\mu} \qquad \langle N \rangle = -\left(\frac{\partial \Omega}{\partial \mu} \right)_{T,V}$$

$$p = -\left(\frac{\partial \Omega}{\partial V} \right)_{T,\mu}$$

$$\langle N \rangle = \sum_f \langle n_f \rangle$$

$$\langle n_f \rangle = \frac{1}{e^{\beta(\varepsilon_f - \mu)} \mp 1} \qquad \begin{matrix} + \text{FD} \\ - \text{BA} \end{matrix}$$

f (kvantni broj) $\longrightarrow$ $(\vec{p}, \vec{r})$ fazni prostor

$$\sum_f \longrightarrow \frac{1}{h^3} \iint d^3\vec{r}\, d^3\vec{p} = \text{(broj mikro-stanja)}$$

$$= \frac{V}{h^3} \int d^3\vec{p}$$

Ako se uzme u obzir i spin $\hat{\vec{s}}$, ima još $g = 2s+1$ stepeni slobode

$$\sum_f \longrightarrow \frac{gV}{h^3} \int d^3\vec{p} \qquad \text{(kontinualna aproksimacija)}$$

1. Pokazati da za idealan nerelativistički gas postoji sledeća relacija, nezavisno od statistike kojoj čestice pripadaju:

$$pV = \frac{2}{3}\langle E\rangle$$

gde je p pritisak sistema, V je zapremina a $\langle E\rangle$ srednja kinetička energija sistema.
(u ovom zadatku je λ const. ; $\lambda = e^{\beta\mu}$)

$$\Xi = \prod_f \left(1 \pm e^{-\beta(\epsilon_f - \mu)}\right)^{\pm 1}$$

$$\Omega = -kT \ln \Xi$$

$$\vdots$$

$$\Omega = \mp kT \sum_f \ln\left(1 \pm e^{-\beta(\epsilon_f - \mu)}\right)$$

F.D. statistika

$$\Omega = -\frac{1}{\beta} \sum_f \ln\left(1 + e^{-\beta(\epsilon_f - \mu)}\right)$$

$f = \{\vec{p}, s\}$, zanemarujemo s (tj. neka je $s=0$)

Kontinualna aproksimacija

$$\sum_{\vec{p}} \longrightarrow \frac{V}{h^3} \int d^3\vec{p}$$

$$\epsilon_{\vec{p}} = \frac{\vec{p}^2}{2m} = \frac{p^2}{2m} = \epsilon_p$$

$$\Omega = -\frac{V}{\beta h^3} \int \ln\left(1+e^{-\beta\left(\frac{\vec{p}^2}{2m}-\mu\right)}\right) d^3\vec{p}$$

$$\Omega = -\frac{4\pi V}{\beta h^3} \int p^2 \ln\left(1+e^{-\beta\left(\frac{p^2}{2m}-\mu\right)}\right) dp$$

$$p = \sqrt{2m\epsilon_p}$$

$$dp = \sqrt{2m}\,\frac{1}{2\sqrt{\epsilon_p}}\,d\epsilon_p = 2^{-\frac{1}{2}} m^{\frac{1}{2}} \epsilon_p^{-\frac{1}{2}}\,d\epsilon_p$$

$$\Omega = -\frac{4\pi V}{\beta h^3} \int_0^\infty 2m\epsilon_p \ln\left(1+e^{-\beta(\epsilon_p-\mu)}\right) 2^{-\frac{1}{2}} m^{\frac{1}{2}} \epsilon_p^{-\frac{1}{2}}\,d\epsilon_p$$

$$\Omega = -\frac{4\pi V}{\beta h^3} \int_0^\infty 2^{\frac{1}{2}} m^{\frac{3}{2}} \epsilon_p^{\frac{1}{2}} \ln\left(1+e^{-\beta(\epsilon_p-\mu)}\right) d\epsilon_p$$

$$\Xi = \sum_{N=0}^\infty \sum_n e^{-\beta(E_{N,n}-\mu N)}$$

$$\Xi = \sum_{N=0}^\infty \sum_n \lambda^N e^{-\beta E_{N,n}}$$

$$\left(\frac{\partial \Xi}{\partial \beta}\right)_{\lambda,V} = \sum_{N=0}^\infty \sum_n \lambda^N (-E_{N,n}) e^{-\beta E_{N,n}}$$

$$\left(\frac{\partial \Xi}{\partial \beta}\right)_{\lambda, V} = -\Xi \langle E \rangle$$

$$\Downarrow$$

$$\langle E \rangle = -\left(\frac{\partial \ln \Xi}{\partial \beta}\right)_{\lambda, V}$$

$$\Omega = -\frac{1}{\beta} \ln \Xi \rightarrow -\ln \Xi = \Omega \cdot \beta$$

$$\langle E \rangle = \left(\frac{\partial \beta \Omega}{\partial \beta}\right)_{\lambda, V}$$

$$\beta \Omega = -VC \int_0^\infty \epsilon_p^{\frac{1}{2}} \ln\left(1 + e^{-\beta(\epsilon_p - \mu)}\right) d\epsilon_p$$

$$= -VC \int_0^\infty \epsilon_p^{\frac{1}{2}} \ln\left(1 + \lambda e^{-\beta \epsilon_p}\right) d\epsilon_p \quad (*)$$

$$\langle E \rangle = \left(\frac{\partial \beta \Omega}{\partial \beta}\right)_{\lambda, V} = -VC \int_0^\infty \frac{\epsilon_p^{\frac{1}{2}} (-\epsilon_p) \lambda e^{-\beta \epsilon_p} d\epsilon_p}{1 + \lambda e^{-\beta \epsilon_p}}$$

$$\langle E \rangle = VC \overset{?}{\lambda} \int_0^\infty \frac{\epsilon_p^{\frac{3}{2}} d\epsilon_p}{1 + \lambda^{-1} e^{\beta \epsilon_p}} \quad (**)$$

Sa druge strane (zbog *)

$$p = -\left(\frac{\partial \Omega}{\partial V}\right)_{T, \mu} = \frac{C}{\beta} \underbrace{\int_0^\infty \epsilon_p^{\frac{1}{2}} \ln\left(1 + \lambda e^{-\beta \epsilon_p}\right) d\epsilon_p}_{I}$$

Domaći: Rešiti integral I metodom parcijalne integracije

$$u = \ln\left(1 + \lambda e^{-\beta \epsilon_p}\right) \qquad dv = \epsilon_p^{\frac{1}{2}} d\epsilon_p$$

Posle parc. int. dobija se

$$\frac{\beta P}{c} = \frac{2}{3}\beta \int_0^\infty \frac{\epsilon_p^{\frac{3}{2}}}{1+\lambda^{-1}e^{\beta\epsilon_p}}\, d\epsilon_p \qquad (***)$$

Na osnovi (**) i (***)

$$pV = \frac{2}{3}\langle E\rangle$$

<u>Domaći</u> ! Za B-A statistiku

Konsultovati udžbenik, str. 132.

12. Претходни задатак али за бозоне!

$$\Xi = \prod_f \left(1 - e^{-\beta(\epsilon_f - \mu)}\right)^{-1}$$

$$\Omega = -kT \ln \Xi$$

$$\Omega = kT \sum_f \ln\left(1 - e^{-\beta(\epsilon_f - \mu)}\right)$$

$f = \{\vec{p}, m_s\}$, $s = 0$, $m_s = 0$, $g = 1$

$$\Omega = \frac{1}{\beta} \sum_{\vec{p}} \ln\left(1 - e^{-\beta(\epsilon_{\vec{p}} - \mu)}\right)$$

$\mu \to 0$ чланови са $\vec{p} = 0$ постају значајни

$$\Omega = \frac{1}{\beta} \ln\left(1 - e^{\beta\mu}\right) + \frac{1}{\beta} \sum_{\vec{p} \neq 0} \ln\left(1 - e^{-\beta(\epsilon_{\vec{p}} - \mu)}\right)$$

$$\lambda = e^{\beta\mu}$$

$$\sum_{\vec{p}} \longrightarrow \frac{V}{h^3} \int d^3\vec{p}$$

$$\beta\Omega = \ln(1-\lambda) + \frac{4\pi V}{h^3} \int_0^\infty \ln\left(1 - \lambda e^{-\beta \frac{p^2}{2m}}\right) p^2 dp$$

$$\langle E \rangle = \left[\frac{\partial(\Omega\beta)}{\partial\beta} \right]_{\lambda, V}$$

$$\langle E \rangle = \lambda V C \int_0^\infty \frac{\epsilon_p^{3/2} \, d\epsilon_p}{e^{\beta\epsilon_p} - \lambda}$$

$$p = -\left(\frac{\partial \Omega}{\partial V} \right)_{T, \mu}$$

$$p = \frac{2}{3} C \lambda \int_0^\infty \frac{\epsilon_p^{3/2} \, d\epsilon_p}{e^{\beta\epsilon_p} - \lambda}$$

$$pV = \frac{2}{3} \langle E \rangle$$

5. Idealan gas od n N bozona nalazi se u zapremini V. Neka N_0 označava broj čestica u najnižem jednočestičnom stanju $\varepsilon_0 = 0\ (\vec{p} = 0)$, a N_1' broj čestica u svim ostalim stanjima $(\vec{p} \neq 0)$. Pokazati da je hemijski potencijal negativna nerastuća f-ja temperature.
(Brojevi čestica su srednje vrednosti)

$$\langle N \rangle = \sum_{\vec{p}} \frac{1}{e^{\beta(\epsilon_{\vec{p}} - \mu)} - 1}$$ ← implicitna f-ja za μ

$\mu = 0 \quad \vec{p} = 0 \Rightarrow \epsilon_p = 0 \Rightarrow \langle N \rangle \to \infty$

$$\langle N \rangle = \frac{1}{e^{-\beta\mu} - 1} + \sum_{\vec{p} \neq 0} \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1}$$

$$\langle N \rangle \equiv N$$

$$N = N(\mu(T), T) = \text{const}$$

$$dN = \frac{\partial N}{\partial \mu} d\mu + \frac{\partial N}{\partial T} dT = 0$$

$$\frac{\partial N}{\partial \mu} \frac{d\mu}{dT} + \frac{\partial N}{\partial T} = 0$$

$$\frac{d\mu}{dT} = - \frac{\frac{\partial N}{\partial T}}{\frac{\partial N}{\partial \mu}} = - \frac{\frac{\partial N}{\partial \beta} \frac{\partial \beta}{\partial T}}{\frac{\partial N}{\partial \mu}} = k\beta^2 \frac{\frac{\partial N}{\partial \beta}}{\frac{\partial N}{\partial \mu}}$$

$$N = N_0 + N_1$$

$$\mu = \mu_0 + \mu_1$$

$$\frac{d\mu}{dT} = \frac{d\mu_0}{dT} + \frac{d\mu_1}{dT}, \text{ tj.}$$

$$\frac{d\mu}{dT} = k\beta^2 \frac{\frac{\partial N_0}{\partial \beta}}{\frac{\partial N_0}{\partial \mu}} + k\beta^2 \frac{\frac{\partial N_1}{\partial \beta}}{\frac{\partial N_1}{\partial \mu}}$$

Za vezbu

$$\frac{\partial N_0}{\partial \beta} = \frac{\mu \, e^{-\beta\mu}}{(e^{-\beta\mu} - 1)^2}$$

$$\frac{\partial N_1}{\partial \beta} = - \sum_{\vec{p} \neq 0} \frac{(\epsilon_p - \mu) \, e^{-\beta(\epsilon_p - \mu)}}{(e^{\beta(\epsilon_p - \mu)} - 1)^2}$$

$$\frac{\partial N_0}{\partial \mu} = \frac{\beta \, e^{-\beta\mu}}{(e^{-\beta\mu} - 1)^2}$$

$$\frac{\partial N_1}{\partial \mu} = \sum_{\vec{p} \neq 0} \frac{\beta \, e^{\beta(\epsilon_p - \mu)}}{(e^{\beta(\epsilon_p - \mu)} - 1)^2}$$

$$\frac{d\mu}{dT} = k\cancel{\beta^2}\,\frac{\dfrac{\mu\,\cancel{e^{-\beta\mu}}}{\cancel{(e^{-\beta\mu}-1)^2}}}{\dfrac{\cancel{\beta}\,\cancel{e^{-\beta\mu}}}{\cancel{(e^{-\beta\mu}-1)^2}}} - k\cancel{\beta^2}\,\frac{\displaystyle\sum_{\vec{p}\neq 0}\frac{(\epsilon_p-\mu)\,e^{\beta(\epsilon_p-\mu)}}{(e^{\beta(\epsilon_p-\mu)}-1)^2}}{\displaystyle\sum_{\vec{p}\neq 0}\frac{\cancel{\beta}\,e^{+\beta(\epsilon_p-\mu)}}{(e^{\beta(\epsilon_p-\mu)}-1)^2}}$$

$$\frac{d\mu}{dT} = \frac{\mu}{T} - \frac{1}{T}\,\frac{\displaystyle\sum_{\vec{p}\neq 0}\frac{(\epsilon_p-\mu)\,e^{\beta(\epsilon_p-\mu)}}{(e^{\beta(\epsilon_p-\mu)}-1)^2}}{\displaystyle\sum_{\vec{p}\neq 0}\frac{e^{+\beta(\epsilon_p-\mu)}}{(e^{\beta(\epsilon_p-\mu)}-1)^2}}$$

TEORIJA! ZA BOZONE $\mu \leq 0$
i prethodni zadatak

$$\frac{d\mu}{dT} < 0$$

1. Pokazati da ako se formula

$$\langle n_f \rangle = \frac{1}{e^{\beta(\epsilon_f - \mu)} \pm 1} \quad (*)$$

može aproksimirati Boltzmann-ovom formulom

$$\langle n_f \rangle \approx e^{-\beta(\epsilon_f - \mu)} \quad (**)$$

da je tada termalna talasna dužina

$$\lambda_T = \frac{h}{\sqrt{2\pi m k T}}$$

identičnih čestica mase m, koje čine idealan gas, mnogo manja od srednjeg rastojanja među česticama.

$(**)$ se dobija iz $(*)$ ako važi

$$\beta(\epsilon_f - \mu) \gg 1$$

odnosno

$$\epsilon_f - \mu(T) \gg kT \quad (***)$$

Pošto su energetski nivoi ϵ_f fiksirani i ne zavise od T, da bi nejednakost $(***)$ važila za $T \gg 0$, hemijski potencijal na tim temperaturama, $\mu(T)$, mora biti negativan, $\mu(T) < 0$ i to tako da važi $|\mu(T)| \gg kT$ odnosno da fugacitet ispunjava uslov

$$\lambda = e^{\frac{\mu(T)}{kT}} \ll 1 \quad (\text{klasično ponašanje})$$

$$\langle N\rangle = \sum_f \langle n_f\rangle$$

Za slobodnu česticu $\quad f \leftrightarrow \vec{p} \quad (s=0)$

$$\langle N\rangle = \sum_{\vec{p}} \langle n_{\vec{p}}\rangle$$

$$\langle N\rangle = \frac{V}{h^3}\int e^{-\beta(\epsilon_{\vec{p}}-\mu)}\, d^3\vec{p}$$

$$\langle N\rangle = \frac{4\pi V}{h^3}\int_0^\infty e^{-\beta\left(\frac{p^2}{2m}-\mu\right)}\, p^2 dp$$

Smena: $\quad \frac{p^2}{2m} = \epsilon_p \Rightarrow p = \sqrt{2m\epsilon_p}$

$$dp = 2^{\frac{1}{2}} m^{\frac{1}{2}} \frac{1}{2\sqrt{\epsilon_p}}\, d\epsilon_p$$

$$dp = 2^{-\frac{1}{2}} m^{\frac{1}{2}} \frac{d\epsilon_p}{\sqrt{\epsilon_p}}$$

$$\langle N\rangle = \frac{4\pi V}{h^3}\int_0^\infty e^{-\beta(\epsilon_p-\mu)}\, 2m\epsilon_p\, 2^{-\frac{1}{2}} m^{\frac{1}{2}} \epsilon_p^{-\frac{1}{2}} d\epsilon_p$$

$$\langle N\rangle = \frac{4\pi V}{h^3} 2^{\frac{1}{2}} m^{\frac{3}{2}} \int_0^\infty \epsilon_p^{\frac{1}{2}} e^{-\beta(\epsilon_p-\mu)}\, d\epsilon_p$$

$$\langle N\rangle = \frac{4\pi V}{h^3} 2^{\frac{1}{2}} m^{\frac{3}{2}} e^{\beta\mu}\int_0^\infty \epsilon_p^{\frac{1}{2}} e^{-\beta\epsilon_p}\, d\epsilon_p$$

Smena $\quad \beta\epsilon_p = \xi$

$$\langle N\rangle = \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} e^{\beta\mu}\int_0^\infty e^{-\xi} \frac{\xi^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\frac{d\xi}{\beta}$$

$$\langle N \rangle = \frac{2\pi V}{h^3} \left(\frac{2m}{\beta}\right)^{\frac{3}{2}} e^{\beta\mu} \underbrace{\int_0^\infty e^{-\xi} \xi^{\frac{1}{2}} d\xi}_{\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}}$$

$$\langle N \rangle = \frac{2\pi V}{h^3} \left(\frac{2m}{\beta}\right)^{\frac{3}{2}} e^{\beta\mu} \frac{\sqrt{\pi}}{2}$$

$$\langle N \rangle = \frac{V}{h^3} \left(\frac{2m\pi}{\beta}\right)^{\frac{3}{2}} e^{\beta\mu}$$

$$\frac{\langle N \rangle}{V} = \frac{e^{\beta\mu}}{h^3} (2\pi m k T)^{\frac{3}{2}} = \frac{e^{\beta\mu}}{\lambda_T^3}$$

$$d^3 = \frac{V}{\langle N \rangle}$$ srednje rastojanje među česticama

$$\frac{1}{d^3} = \frac{e^{\beta\mu}}{\lambda_T^3} \Rightarrow \frac{\lambda_T^3}{d^3} = e^{\beta\mu} \ll 1$$ (Klasično ponašanje

odnosno

$$\boxed{\lambda_T \ll d}$$

5. Odrediti fluktuacije broja čestica idealnog kvantnog gasa koje zadovoljavaju: (u TD ravnoteži)

a) Fermi-Dirac-ovu

b) Bose-Einstein-ovu statistiku.

tako da budu izražene preko srednjeg broja popunjenosti jednočestičnih nivoa

$$D(N) = \langle N^2 \rangle - \langle N \rangle^2$$

$$\langle N^2 \rangle = \frac{\sum_{N=0}^{\infty} \sum_{n} N^2 e^{-\beta(E_{N,n} - \mu N)}}{\Xi}$$

$$\langle N^2 \rangle = \frac{1}{\Xi} \frac{1}{\beta^2} \frac{\partial^2}{\partial \mu^2} \sum_N \sum_n e^{-\beta(E_{N,n} - \mu N)}$$

$$\langle N \rangle = \frac{1}{\Xi} \sum_N \sum_n N e^{-\beta(E_{N,n} - \mu N)}$$

$$\langle N \rangle = \frac{1}{\Xi} \frac{1}{\beta} \frac{\partial}{\partial \mu} \sum_N \sum_n e^{-\beta(E_{N,n} - \mu N)}$$

odnosno

$$\langle N^2 \rangle = \frac{1}{\Xi} \frac{1}{\beta^2} \frac{\partial^2 \Xi}{\partial \mu^2} \quad \text{i}$$

$$\langle N \rangle = \frac{1}{\Xi} \frac{1}{\beta} \frac{\partial \Xi}{\partial \mu}$$

$$D(N) = \frac{1}{\Xi}\frac{1}{\beta^2}\frac{\partial^2 \Xi}{\partial \mu^2} - \frac{1}{\beta^2}\frac{1}{\Xi^2}\left(\frac{\partial \Xi}{\partial \mu}\right)^2$$

$$D(N) = \frac{1}{\beta^2}\frac{\Xi \frac{\partial^2 \Xi}{\partial \mu^2} - \left(\frac{\partial \Xi}{\partial \mu}\right)^2}{\Xi^2}$$

$$= \frac{1}{\beta^2}\frac{\partial}{\partial \mu}\left(\frac{1}{\Xi}\frac{\partial \Xi}{\partial \mu}\right)$$

$$= \frac{1}{\beta^2}\frac{\partial}{\partial \mu}\left(\beta \langle N \rangle\right)$$

$$= kT \frac{\partial \langle N \rangle}{\partial \mu}$$

2) $$\langle n_f \rangle = \frac{1}{e^{\beta(\epsilon_f - \mu)} + 1}$$

$$\langle N \rangle = \sum_f \langle n_f \rangle = \sum_f \frac{1}{e^{\beta(\epsilon_f - \mu)} + 1}$$

$$\frac{\partial \langle N \rangle}{\partial \mu} = \sum_f - \frac{\left(e^{\beta(\epsilon_f - \mu)} + 1\right)'_\mu}{\left(e^{\beta(\epsilon_f - \mu)} + 1\right)^2}$$

$$= - \sum_f \frac{(-\beta)\, e^{\beta(\epsilon_f - \mu)}}{\left(e^{\beta(\epsilon_f - \mu)} + 1\right)^2}$$

$$\frac{\partial \langle N \rangle}{\partial \mu} = \beta \sum_f \frac{e^{\beta(\epsilon_f - \mu)}}{\left(e^{\beta(\epsilon_f - \mu)} + 1\right)^2}$$

$$= \beta \sum_f \left(\frac{e^{\beta(\epsilon_f - \mu)} + 1}{\left(e^{\beta(\epsilon_f - \mu)} + 1\right)^2} - \frac{1}{\left(e^{\beta(\epsilon_f - \mu)} + 1\right)^2} \right)$$

$$= \beta \left(\sum_f \frac{1}{e^{\beta(\epsilon_f - \mu)} + 1} - \sum_f \frac{1}{\left(e^{\beta(\epsilon_f - \mu)} + 1\right)^2} \right)$$

$$= \beta \sum_f \left(\langle n_f \rangle - \langle n_f \rangle^2 \right)$$

$$D(N) = kT \beta \sum_f \langle n_f \rangle \left(1 - \langle n_f \rangle\right)$$

$$D(N) = \sum_f \langle n_f \rangle \left(1 - \langle n_f \rangle\right)$$

b) Domaći

$$D(N) = \sum_f \langle n_f \rangle \left(1 + \langle n_f \rangle\right)$$

8. Za kvantni idealni gas naći izraz za entropiju, izraženog preko srednjeg broja popunjenosti jednočestičnih energetskih nivoa.

Raditi za bozone ili fermione!

$$\langle n_f \rangle = \frac{1}{e^{\beta(\epsilon_f - \mu)} \pm 1}$$

$+$ F.D.

$-$ B.A.

$$\Xi = \prod_f \left(1 \pm e^{-\beta(\epsilon_f - \mu)}\right)^{\pm 1}$$

$$\Omega = -kT \ln \Xi$$

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu}$$

$$\Omega = \mp kT \ln \prod_f \left(1 \pm e^{-\beta(\epsilon_f - \mu)}\right)$$

$$= \mp \frac{1}{\beta} \sum_f \ln \left(1 \pm e^{-\beta(\epsilon_f - \mu)}\right)$$

$$S = -\frac{\partial \Omega}{\partial T} = -\frac{\partial \Omega}{\partial \beta}\frac{\partial \beta}{\partial T} = k\beta^2 \frac{\partial \Omega}{\partial \beta}$$

$$S = k\beta^2 \left\{ \frac{\partial}{\partial \beta} \left(\mp \frac{1}{\beta} \sum_f \ln \left(1 \pm e^{-\beta(\epsilon_f - \mu)}\right)\right)\right.$$

$$S = \mp K\beta^2 \left\{ -\frac{1}{\beta^2} \sum_f \ln\left(1 \pm e^{-\beta(\epsilon_f - \mu)}\right) + \right.$$

$$\left. \frac{1}{\beta} \sum_f \frac{1}{1 \pm e^{-\beta(\epsilon_f - \mu)}} \left(1 \pm e^{-\beta(\epsilon_f - \mu)}\right)'_\beta \right\}$$

$$S = \mp K \left\{ -\sum_f \ln\left(1 \pm e^{-\beta(\epsilon_f - \mu)}\right) + \beta \times \right.$$

$$\left. \times \sum_f \frac{1}{1 \pm e^{-\beta(\epsilon_f - \mu)}} \left(\pm e^{-\beta(\epsilon_f - \mu)}\left(-(\epsilon_f - \mu)\right)\right) \right\}$$

$$S = \mp K \left\{ -\sum_f \ln \frac{e^{\beta(\epsilon_f - \mu)} \pm 1}{e^{\beta(\epsilon_f - \mu)}} \mp \right.$$

$$\left. \beta \sum_f \frac{(\epsilon_f - \mu)\, e^{-\beta(\epsilon_f - \mu)}}{1 \pm e^{-\beta(\epsilon_f - \mu)}} \right\}$$

$\boxed{\text{Domácí}}$

$$\langle n_f \rangle = \frac{1}{e^{\beta(\epsilon_f - \mu)} \pm 1}$$

$$\Downarrow$$

$$\beta(\epsilon_f - \mu) = \ln\left(\frac{1 \mp \langle n_f \rangle}{\langle n_f \rangle}\right)$$

$$S = \mp k \left\{ - \sum_f \ln \frac{e^{\beta(\epsilon_f-\mu)} \pm 1}{e^{\beta(\epsilon_f-\mu)}} \mp \beta \sum_f \frac{\epsilon_f-\mu}{e^{\beta(\epsilon_f-\mu)} \pm 1} \right\}$$

$$S = \mp k \left\{ - \sum_f \ln \langle n_f \rangle^{-1} + \sum_f \ln e^{\beta(\epsilon_f-\mu)} \mp \beta \sum_f (\epsilon_f-\mu) \langle n_f \rangle \right\}$$

$$S = \mp K \left\{ \sum_f \ln \langle n_f \rangle + \sum_f \beta(\epsilon_f-\mu) \mp \beta \sum_f (\epsilon_f-\mu) \langle n_f \rangle \right\}$$

$$S = \mp K \left\{ \sum_f \ln \langle n_f \rangle + \sum_f \ln \left(\frac{1 \mp \langle n_f \rangle}{\langle n_f \rangle} \right) \mp \sum_f \ln \left(\frac{1 \mp \langle n_f \rangle}{\langle n_f \rangle} \right) \langle n_f \rangle \right\}$$

$$S = \mp K \left\{ \cancel{\sum_f \ln \langle n_f \rangle} + \sum_f \ln (1 \mp \langle n_f \rangle) - \cancel{\sum_f \ln \langle n_f \rangle} \mp \sum_f \langle n_f \rangle \ln (1 \mp \langle n_f \rangle) \pm \sum_f \langle n_f \rangle \ln \langle n_f \rangle \right\}$$

$$S = \mp K \left\{ \sum_f \ln\left(1 \mp \langle n_f \rangle\right) \mp \sum_f \langle n_f \rangle \ln\left(1 \mp \langle n_f \rangle\right) \pm \sum_f \langle n_f \rangle \ln \langle n_f \rangle \right\}$$

$$S = \overset{\mp}{+} \frac{K}{\beta^4} \left\{ \sum_f \left(1 \mp \langle n_f \rangle\right) \ln\left(1 \mp \langle n_f \rangle\right) \pm \sum_f \langle n_f \rangle \ln \langle n_f \rangle \right\}$$

Gornji znak F.D., a donji B.A. stat.

Изгледа да фали β^4

Za idealni fotonski gas, koji se nalazi u šupljini, naći srednji broj fotona u jediničnoj zapremini, unutrašnju energiju po jedinici zapremine i gustinu energije po jediničnom intervalu frekvencije.

---

1. EM polje u šupljini modelujemo fotonskim gasom → polje kao diskretan sistem
2. Fotoni ne interaguju jedni sa drugima
3. Broj fotona u šupljini $\boxed{\text{nije konstantan}}$ je neodređen
4. Zbog 2., fotonskom gasu je potrebna interakcija sa supstancom (zid šupljine)
5. Efektivni spin fotona je $\frac{1}{2}$ (sledi iz relativističke kvantne fizike)
6. Hemijski potencijal fotonskog gasa je nula

$$\underbrace{\epsilon = cp \qquad \epsilon = \hbar\omega}$$

$$p = \frac{\hbar\omega}{c}, \quad \vec{p} = \hbar\vec{k}, \quad k = \frac{\omega}{c}$$

$$s = \frac{\hbar}{2}$$

$$g = 2s+1$$
$$= 2$$

$$\langle n_{\vec{p}} \rangle = \frac{1}{e^{\beta c p} - 1}$$

$$\langle N \rangle = \sum_{m_s} \sum_{\vec{p}} \langle n_{\vec{p}} \rangle = g \sum_{\vec{p}} \langle n_{\vec{p}} \rangle$$

$$= 2 \frac{V}{h^3} \int \frac{d^3\vec{p}}{e^{\beta c p} - 1}$$

Domaći

Pokazati da važi

$$\frac{V}{h^3} \int d^3\vec{p} = \frac{V}{(2\pi)^3} \int d^3\vec{k}$$

$\vec{k}$ – talasni vektor

Broj fotona u jediničnoj zapremini

$$\frac{\langle N \rangle}{V} = \frac{2}{(2\pi)^3} \int_0^\infty \frac{4\pi k^2}{e^{\beta \hbar c k} - 1} dk$$

$$k = \frac{\omega}{c}$$

$$\frac{\langle N \rangle}{V} = \frac{8\pi}{(2\pi)^3} \int_0^\infty \frac{\frac{\omega^2}{c^2}}{e^{\beta \hbar \omega} - 1} \frac{d\omega}{c}$$

$$\frac{\langle N\rangle}{V} = \frac{1}{\pi^2 c^3}\int_0^\infty \frac{\omega^2}{e^{\beta\hbar\omega}-1}\,d\omega$$

Smena $\beta\hbar\omega = x$

$$d\omega = \frac{dx}{\beta\hbar}$$

$$\frac{\langle N\rangle}{V} = \frac{1}{\pi^2 c^3}\int_0^\infty \frac{\frac{x^2}{(\beta\hbar)^2}}{e^x-1}\,\frac{dx}{\beta\hbar}$$

$$\frac{\langle N\rangle}{V} = \frac{1}{\pi^2 c^3}\,\frac{1}{(\beta\hbar)^3}\underbrace{\int_0^\infty \frac{x^2}{e^x-1}\,dx}_{I}$$

I je kv. integral. Po kom kriterijumu?

$$\frac{\langle N\rangle}{V} = \frac{I}{\pi^2 c^3}\,\frac{1}{\left(\frac{h}{2\pi k T}\right)^3} = \frac{I}{\pi^2 c^3}\,\frac{2^3\pi^3 k^3 T^3}{h^3}$$

$$= \frac{8 I \pi k^3}{c^3 h^3}\,T^3$$

Domaći zadatak

Na osnovi integrala (7.17) i tabele 7.1 nađi vrednost integrala I

Unutrašnja energija je dana kao

$$U = 2\sum_{\vec{p}} \epsilon_{\vec{p}} \langle n_{\vec{p}} \rangle = 2\frac{V}{\hbar^3}\int d^3\vec{p}\; pc \frac{1}{e^{\beta pc}-1}$$

$$= \frac{2Vc}{\hbar^3}\cdot 4\pi \int_0^\infty \frac{p^3\,dp}{e^{\beta pc}-1} \quad ; \quad p = \frac{\hbar\omega}{c}$$

$$= \frac{8\pi V c}{\hbar^3}\int_0^\infty \frac{\frac{\hbar^3\omega^3}{c^3}\frac{\hbar}{c}\,d\omega}{e^{\beta\hbar\omega}-1}$$

$$= \frac{8\pi V c}{\hbar^3}\frac{\hbar^4}{c^4}\int_0^\infty \frac{\omega^3\,d\omega}{e^{\beta\hbar\omega}-1} \quad (*) \quad ; \quad \beta\hbar\omega = x, \quad d\omega = \frac{dx}{\beta\hbar}$$

$$= \frac{8\pi V c}{\hbar^3}\frac{\hbar^4}{c^4}\int_0^\infty \frac{\frac{x^3}{\beta^3\hbar^3}}{e^x-1}\frac{dx}{\beta\hbar}$$

$$= \frac{8\pi V I}{c^3\hbar^3}k^4 T^4$$

$$\frac{U}{V} = \frac{8\pi I k^4}{c^3\hbar^3}T^4$$

$$u = \sigma T^4$$ (Stefan-Boltzmann-ov zakon)

Vratimo se par koraka nazad

$$\frac{U}{V} = \frac{8\pi c}{h^3}\frac{\hbar^4}{c^4}\int_0^\infty \frac{\omega^3 \, d\omega}{e^{\beta\hbar\omega}-1} \longleftarrow$$

$$\frac{U}{V} = \frac{\hbar}{2\pi^3 c^3}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega$$

$$\frac{U}{V} = \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega$$

$$\frac{U}{V} = \int_0^\infty u(\omega, T)\, d\omega$$

Gustina energije po jediničnom intervalu frekvencije

$$\underbrace{u(\omega, T) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\beta\hbar\omega}-1}}$$

Planck-ova raspodela
zakon zračenja

Domaći zadatak

$$u(\omega, T) \longrightarrow u(\lambda, T)$$

$\omega = 2\pi\nu \qquad \nu = \frac{c}{\lambda}$

Pokazati kako glase granični slučajevi

a) $\hbar\omega \ll kT$ (Rayleigh–Jeans-ova f-la)

b) $\hbar\omega \gg kT$ (Wien-ova f-la)

8. Pokazati da je unutrašnja energija idealnog Bose-ovog gasa data izrazom

$$\mathcal{U} = \frac{3}{2} kTV \left(\frac{2\pi m kT}{h^2}\right)^{\frac{3}{2}} \sum_{l=1}^{\infty} \frac{e^{\beta l \mu}}{l^{5/2}}$$

kada je degeneracija stanja.

$$U = \langle E \rangle = \sum_{\vec{p}} \varepsilon_{\vec{p}} \langle n_{\vec{p}} \rangle \quad , \quad g = 1$$

$$= \sum_{\vec{p}} \frac{\epsilon_{\vec{p}}}{e^{\beta(\epsilon_{\vec{p}} - \mu)} - 1}$$

Kontinualna aproksimacija

$$U = \frac{V}{h^3} \int \frac{\epsilon_{\vec{p}}}{e^{\beta(\epsilon_{\vec{p}} - \mu)}} d^3\vec{p}$$

$$\epsilon_{\vec{p}} = \frac{\vec{p}^2}{2m} = \frac{p^2}{2m} = \epsilon$$

$$U = \frac{V}{h^3} 4\pi \int_0^{\infty} \frac{\frac{\vec{p}^2}{2m}}{e^{\beta(\frac{\vec{p}^2}{2m} - \mu)} - 1} p^2 dp$$

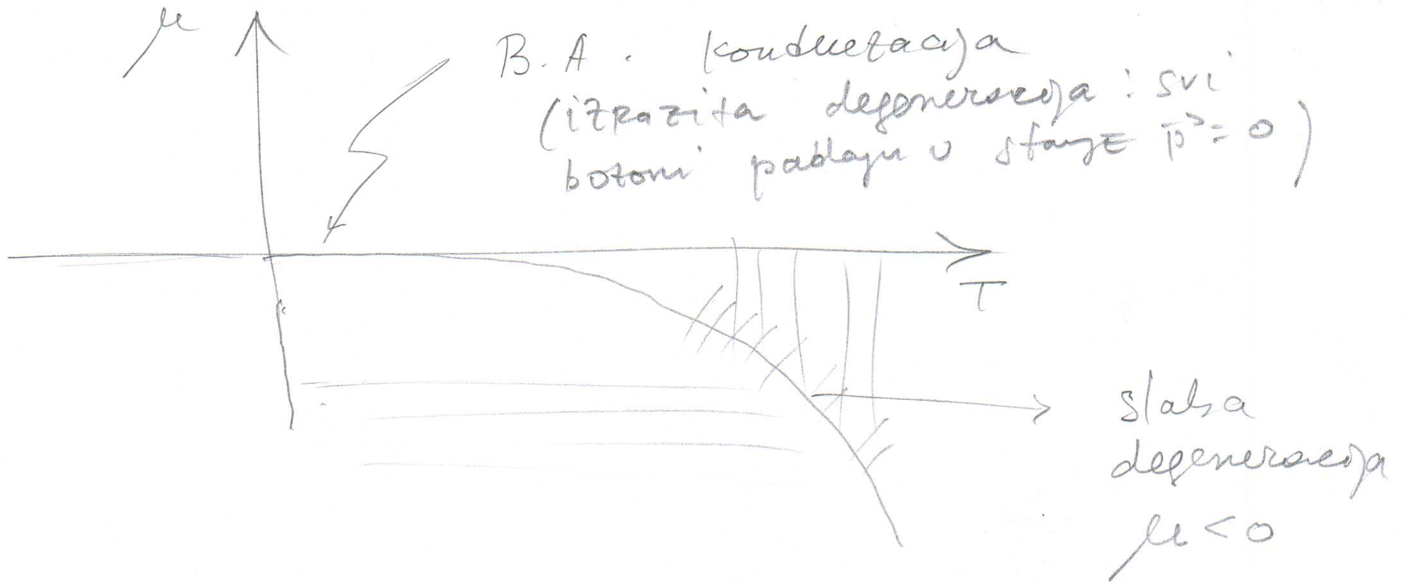
$$\epsilon = \frac{p^2}{2m} \longrightarrow p = \sqrt{2m\epsilon}$$

$$dp = \sqrt{2m} \frac{d\epsilon}{2\sqrt{\epsilon}} = 2^{-\frac{1}{2}} m^{\frac{1}{2}} \frac{d\epsilon}{\sqrt{\epsilon}}$$

$$U = \frac{V}{h^3} 4\pi \int_0^{\infty} \frac{\epsilon}{e^{\beta(\epsilon - \mu)} - 1} 2m\epsilon \, 2^{-\frac{1}{2}} m^{\frac{1}{2}} \frac{d\epsilon}{\sqrt{\epsilon}}$$

$$U = \frac{V}{h^3} 4\pi \, 2^{\frac{1}{2}} m^{\frac{3}{2}} \int_0^\infty \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}-1} d\epsilon$$

$$U = \frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \int_0^\infty \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}-1} d\epsilon$$



$$e^{-\beta(\epsilon-\mu)} \ll 1$$

Inverzno razmišljanje u odnosu na primenu geometrijskog reda

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

Ali,
$$\frac{1}{1-x} = 1 + \sum_{n=1}^{\infty} x^n$$

$$\frac{1}{1-x} - 1 = \sum_{n=1}^{\infty} x^n$$

$$\frac{x}{\cancel{1}-x} = \sum_{n=1}^{\infty} x^n$$

$$\frac{1}{e^{\beta(\epsilon-\mu)}-1} = \frac{e^{-\beta(\epsilon-\mu)}}{1-e^{-\beta(\epsilon-\mu)}}$$

Danke,

$$\frac{e^{-\beta(\epsilon-\mu)}}{1-e^{-\beta(\epsilon-\mu)}} = \sum_{\ell=1}^{\infty} \left(e^{-\beta(\epsilon-\mu)}\right)^{\ell}$$

$$= \sum_{\ell=1}^{\infty} e^{\ell\beta(\mu-\epsilon)}$$

$$U = \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} \int_0^{\infty} \epsilon^{\frac{3}{2}} \sum_{\ell=1}^{\infty} e^{\ell\beta(\mu-\epsilon)}\, d\epsilon$$

$$U = \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} \sum_{\ell=1}^{\infty} \int_0^{\infty} \epsilon^{\frac{3}{2}} e^{\ell\beta(\mu-\epsilon)}\, d\epsilon$$

$$U = \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} \sum_{\ell=1}^{\infty} e^{\ell\beta\mu} \int_0^{\infty} \epsilon^{\frac{3}{2}} e^{-\beta\ell\epsilon}\, d\epsilon$$

$$\beta\ell\epsilon = x$$
$$d\epsilon = \frac{dx}{\beta\ell}$$

$$U = \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} \sum_{\ell=1}^{\infty} e^{\ell\beta\mu} \int_0^{\infty} \frac{x^{\frac{3}{2}}}{(\beta\ell)^{\frac{3}{2}}} e^{-x} \frac{dx}{\beta\ell}$$

$$U = \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} \sum_{\ell=1}^{\infty} \frac{e^{\ell\beta\mu}}{\beta^{5/2}\ell^{5/2}} \underbrace{\int_0^{\infty} x^{\frac{3}{2}} e^{-x}\, dx}_{\frac{3}{4}\sqrt{\pi}}$$

$$U = \frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} (kT)^{\frac{3}{2}} \cdot \frac{3}{4}\sqrt{\pi} \sum_{l=1}^{\infty} \frac{e^{l\beta\mu}}{l^{5/2}}$$

$$U = \frac{3}{2} kT \frac{(2m\pi kT)^{\frac{3}{2}}}{h^3} \sum_{l=1}^{\infty} \frac{e^{l\beta\mu}}{l^{5/2}}$$

Za pismeni deo ispita varijanta sa fotonskim gasom

$$U = \frac{3}{2} kTV \left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} \sum_{l=1}^{\infty} \frac{e^{\beta l\mu}}{l^{5/2}} \quad (\mu = 0)$$

7. Stanje TD ravnoteže kristalne rešetke od N atoma, koja osciluje, približno je ekvivalentno stanju ravnoteže od 3N neinteragujućih 1D kvantno-mehaničnih oscilatora. Prema tzv. Debye-evoj aproksimaciji, postoji

$$dn_\omega = \begin{cases} \frac{9N}{\omega_D^3} \omega^2 d\omega & \omega < \omega_D \\ 0 & \omega > \omega_D \end{cases}$$

oscilatora sa frekvencijama između ω i $\omega + d\omega$. Konstanta ω_D, tzv. Debye-eva učestanost, određena je uslovom

$$\int_0^{\omega_D} dn_\omega = 3N$$

Prihvatajući ovu aproksimaciju, odrediti toplotni kapacitet kristalne rešetke pri veoma niskim i pri veoma visokim temperaturama.

Srednja energija kvantnog LHO-a

$$\langle \epsilon \rangle_{LHO} = \hbar\omega \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} \right)$$

$$\langle E \rangle = \int_0^\infty \langle \epsilon(\omega) \rangle \, dn_\omega$$

$$\langle E \rangle = \int_0^\infty \left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} \right) dn_\omega$$

$$\langle E \rangle = \int_0^{\omega_D} \left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} \right) \frac{9N}{\omega_D^3} \omega^2 d\omega$$

$$\langle E \rangle = \frac{9N}{\omega_D^3} \int_0^{\omega_D} \left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} \right) \omega^2 d\omega$$

$$C_V = \left(\frac{\partial \langle E \rangle}{\partial T} \right)_V = \frac{\partial \langle E \rangle}{\partial \beta} \frac{\partial \beta}{\partial T} = -k\beta^2 \frac{\partial \langle E \rangle}{\partial \beta}$$

$$\langle E \rangle = \frac{9N}{\omega_D^3} \int_0^{\omega_D} \left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \right) \omega^2 d\omega$$

$$C_V = -k\beta^2 \frac{\partial}{\partial \beta} \left(\frac{9N}{\omega_D^3} \int_0^{\omega_D} \frac{\hbar\omega^3}{e^{\beta\hbar\omega} - 1} d\omega \right)$$

$$C_V = -k\beta^2 \frac{9N\hbar}{\omega_D^3} \frac{\partial}{\partial \beta} \left(\int_0^{\omega_D} \frac{\omega^3 d\omega}{e^{\beta\hbar\omega} - 1} \right)$$

$$C_V = -k\beta^2 \frac{9N\hbar}{\omega_D^3} \int_0^{\omega_D} \left(- \frac{\hbar\omega e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2} \right) \omega^3 d\omega$$

$$C_V = k\beta^2 \frac{9N\hbar^2}{\omega_D^3} \int_0^{\omega_D} \frac{\omega^4 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2} d\omega$$

Uvedimo smenu

$$\beta\hbar\omega = x$$

$$d\omega = \frac{dx}{\hbar\beta}$$

$$\omega \in [0, \omega_D] \longrightarrow x \in [0, \beta\hbar\omega_D]$$

$$\beta\hbar\omega_D = \frac{\hbar\omega_D}{kT} = \frac{T_D}{T} \; ; \quad T_D = \frac{\hbar\omega_D}{k}$$

Debye-eva temperatura (zavisi od geometrijskih svojstava rešetke)

$$C_V = k\beta^2 \frac{9N\hbar^2}{\omega_D^3} \int_0^{T_D/T} \frac{\frac{x^4}{\beta^4\hbar^4}\, e^x}{(e^x-1)^2} \frac{dx}{\beta\hbar}$$

$$C_V = k\cancel{\beta^2}\cancel{\hbar^2} \frac{9N}{\omega_D^3} \frac{1}{\beta^5\hbar^5} \int_0^{T_D/T} \frac{x^4 e^x}{(e^x-1)^2}\, dx$$

$$C_V = \frac{9Nk}{\omega_D^3 \beta^3 \hbar^3} \int_0^{T_D/T} \frac{x^4 e^x}{(e^x-1)^2}\, dx$$

<u>Na niskim temperaturama</u> $\quad T \ll T_D$

pa ćemo $\int_0^{T_D/T} \xrightarrow{\sim} \int_0^{\infty}$

$$\int_0^{\infty} \frac{x^4 e^x}{(e^x-1)^2} dx = \frac{4\pi^4}{15}$$

$$C_V = 9NK \cdot \frac{1}{\omega_D^3 \frac{1}{K^3T^3} \hbar^3} \frac{4\pi^4}{15}$$

$$C_V = 9NK \left(\frac{T}{T_D}\right)^3 \frac{4\pi^4}{15}$$

$$C_V = 3NK \left(\frac{T}{T_D}\right)^3 \frac{4\pi^4}{5} \sim T^3$$

<u>Na visokim temperaturama</u> $T \gg T_D$

$$x = \beta\hbar\omega = \frac{\hbar\omega}{KT} \to 0$$

$$\frac{x^4 e^x}{(e^x-1)^2} \approx \frac{x^4\left(1+x+\frac{x^2}{2}+\ldots\right)}{\left(1+x+\frac{x^2}{2}+\ldots-1\right)^2} \to O(x^2)$$

$$\approx \frac{x^4(1+x)}{x^2(1+x)^2} \sim x^2$$

$$\frac{x^4 (1+x)}{x^2 (1+x)^2}$$

$$C_V = 9NK \left(\frac{T}{T_D}\right)^3 \int_0^{T_D/T} x^2 dx \qquad \frac{x^2}{1+x} \sim x^2$$

$C_V = 3NK \leftarrow$ Dulong-Petit-ov zakon

$\downarrow$

$\boxed{\text{Fizika čvrstog stanja!}}$

Što niža temperatura, Debye-ov model tačnije reprodukuje toplotno ponašanje kristala!

Koristeći formalizam kanonskog ansambla, naći srednju vrednost popunjenosti jednočestičnih energetskih nivoa za idealni kvantni bozonski* sistem gas sa neodređenim brojem čestica** koje slabo interaguju, a van su polja spoljašnjih sila. (** tj. broj čestica u sistemu nije stalan)

Ovde se podrazumeva da je u pitanju kvantni sistem

$$Z = \sum_{\{E_n\}} g(E_n)\, e^{-\beta E_n}$$

$$\begin{pmatrix} n_1 & n_2 & \dots & n_f & \dots \\ \varepsilon_1 & \varepsilon_2 & & \varepsilon_f & \dots \end{pmatrix}$$

$$E_n = \sum_f n_f \varepsilon_f \,, \qquad \sum_f n_f = N$$

Za kvantne (idealne) sisteme $g(E_n) = 1$, a zbog $E_n = \sum_f n_f \varepsilon_f$, pišemo

$$Z = \sum_{\{n_f\}} e^{-\beta \sum_f n_f \varepsilon_f}$$

114. str. Živić

Srednja vrednost ~~...~~ broja čestica u stanju τ je po definiciji

$$\langle n_\tau \rangle = \frac{\sum_{n_1, n_2 \dots} n_\tau e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \dots + n_\tau\varepsilon_\tau \dots)}}{\sum_{n_1, n_2, \dots} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \dots n_\tau\varepsilon_\tau + \dots)}}$$

Ako ne važi uslov $\sum_f n_f = N$ (za fotonski gas, npr.), onda gornje granice u sumama nisu korelisane. Gornji izraz se može prepisati

$$\langle n_\tau \rangle = \frac{\left(\sum_{n_\tau} n_\tau e^{-\beta\varepsilon_\tau n_\tau}\right)\left(\sum^{(\tau)}_{n_1, n_2, \dots} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \dots)}\right)}{\left(\sum_{n_\tau} e^{-\beta\varepsilon_\tau n_\tau}\right)\left(\sum^{(\tau)}_{n_1, n_2, \dots} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \dots)}\right)}$$

i u slučaju da ne važi $\sum_f n_f = N$, članovi sa $\sum^{(\tau)}_{n_1, n_2, \dots}$ (sumiranje po svim stanjima, osim po τ)

se skraćuju. Zato je

$$\langle n_\tau \rangle = \frac{\sum_{n_\tau = 0}^{\infty} n_\tau e^{-\beta\varepsilon_\tau n_\tau}}{\sum_{n_\tau = 0}^{\infty} e^{-\beta\varepsilon_\tau n_\tau}}$$

Ako je $Z=\sum_{n_r=0}^{\infty} e^{-\beta \epsilon_r n_r}$, onda je

$$\frac{\partial Z}{\partial(\beta\epsilon_r)} = -\sum_{n_r=0}^{\infty} n_r e^{-\beta\epsilon_r n_r}$$

$$\frac{1}{Z}\frac{\partial Z}{\partial(\beta\epsilon_r)} = -\frac{1}{Z}\sum_{n_r=0}^{\infty} n_r e^{-\beta\epsilon_r n_r}$$

$$\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_r} = -\langle n_r\rangle$$

$$\langle n_r\rangle \equiv -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_r}$$

Zapišimo Z malo drugačije

$$Z = \sum_{n_r=0}^{\infty}\left(e^{-\beta\epsilon_r}\right)^{n_r}$$

$$\beta\epsilon_r \equiv x \Rightarrow x>0, \forall x \Rightarrow |e^{-x}|<1$$

Ispunjeni je uslov za geom. red.

$$Z = \frac{1}{1-e^{-\beta\epsilon_r}}$$

$$Z = \left(1-e^{-\beta\epsilon_r}\right)^{-1} \Rightarrow \ln Z = -\ln\left(1-e^{-\beta\epsilon_r}\right)$$

$$-\langle n_r\rangle = -\frac{1}{\beta}\frac{\partial}{\partial \varepsilon_r}\left(-\ln\left(1-e^{-\beta\varepsilon_r}\right)\right)$$

$$= \frac{1}{\beta}\frac{\partial}{\partial \varepsilon_r}\left(\ln\left(1-e^{-\beta\varepsilon_r}\right)\right)$$

$$= \frac{1}{\beta}\frac{1}{1-e^{-\beta\varepsilon_r}}\beta e^{-\beta\varepsilon_r} = \frac{e^{-\beta\varepsilon_r}}{1-e^{-\beta\varepsilon_r}}$$

$$\langle n_r\rangle = \frac{1}{e^{\beta\varepsilon_r}-1}$$

Fotonski gas

$\varepsilon^2 = p^2c^2 + m_0^2c^4$; za foton $m_0 = 0$

$\varepsilon = pc$, $p = \frac{h}{\lambda}$

$$\varepsilon = \frac{ch}{\lambda} = h\nu = \hbar\omega$$

$$\boxed{\varepsilon_r = \hbar\omega_r}$$

$$\langle n_r\rangle = \frac{1}{e^{\beta\hbar\omega_r}-1}$$

~~Plankova raspodela za fotone!~~

Pogledati